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Operator upper bounds for Davis-Choi-Jensen's difference in Hilbert spaces

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ABSTRACT. In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f: I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples of convex and operator convex functions are also provided.

1. INTRODUCTION

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [9, p. 18]) we can introduce the following definition.

Definition 1. A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalized if it preserves the identity operator, i.e., $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

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We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$.
If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalized.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [9, p. 22]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{2}$$

Let $C_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with

$$\sum_{j=1}^{k} C_j^* C_j = 1_H. \tag{3}$$

The map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by [9, p. 19]

$$\Phi(A) := \sum_{j=1}^{k} C_j^* A C_j$$

is a normalized positive linear map on $\mathcal{B}(H)$.

For more results on inequlities for selfadjoint operators in Hilbert spaces, see [2, 3, 6–8] and the references therein.

In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples of convex and operator convex functions are also provided.

2. Main results

We use the following result that was obtained in [4].

Lemma 1. *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then*

$$
\begin{aligned}
(4)\qquad 0 &\le \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \\
&\le (b-t)(t-a) \frac{f'_-(b) - f'_+(a)}{b-a} \le \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right]
\end{aligned}
$$

for any $t \in [a, b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f : [m, M] \to \mathbb{R}$ be a convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A) \subset [m, M]$.*

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$
\begin{aligned}
(5)\qquad &\Phi(f(A)) - f(\Phi(A)) \\
&\le \frac{f'_-(M) - f'_+(m)}{M-m} (M 1_K - \Phi(A))(\Phi(A) - m 1_K) \\
&\le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_K.
\end{aligned}
$$

Proof. Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \le T \le 1_H$ and the convexity of f on $[m, M]$, we have

$$
(6)\qquad f(m(1_H - T) + MT) \le f(m)(1_H - T) + f(M) T
$$

in the operator order.

If we take in (6)

$$
0 \le T = \frac{A - m1_H}{M-m} \le 1_H,
$$

then we get

$$
\begin{aligned}
(7)\qquad & f\left(m\left(1_H - \frac{A - m1_H}{M-m}\right) + M\frac{A - m1_H}{M-m}\right) \\
&\le f(m)\left(1_H - \frac{A - m1_H}{M-m}\right) + f(M)\frac{A - m1_H}{M-m}.
\end{aligned}
$$

Observe that

$$m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m}$$
$$=\frac{m\left(M1_H-A\right)+M\left(A-m1_H\right)}{M-m}=A$$

and

$$f\left(m\right)\left(1_H-\frac{A-m1_H}{M-m}\right)+f\left(M\right)\frac{A-m1_H}{M-m}$$
$$=\frac{f\left(m\right)\left(M1_H-A\right)+f\left(M\right)\left(A-m1_H\right)}{M-m}$$

and by (7) we get the following inequality of interest

$$f\left(A\right)\leq\frac{f\left(m\right)\left(M1_H-A\right)+f\left(M\right)\left(A-m1_H\right)}{M-m}. \tag{8}$$

If we take the map Φ in (8), then we get

$$\begin{aligned}\Phi\left(f\left(A\right)\right)&\leq\Phi\left[\frac{f\left(m\right)\left(M1_H-A\right)+f\left(M\right)\left(A-m1_H\right)}{M-m}\right]\\&=\frac{f\left(m\right)\Phi\left(M1_H-A\right)+f\left(M\right)\Phi\left(A-m1_H\right)}{M-m}\\&=\frac{f\left(m\right)\left(M\Phi\left(1_H\right)-\Phi\left(A\right)\right)+f\left(M\right)\left(\Phi\left(A\right)-m\Phi\left(1_H\right)\right)}{M-m}\\&=\frac{f\left(m\right)\left(M1_K-\Phi\left(A\right)\right)+f\left(M\right)\left(\Phi\left(A\right)-m1_K\right)}{M-m},\end{aligned}$$

which implies that

$$\begin{aligned}&\Phi\left(f\left(A\right)\right)-f\left(\Phi\left(A\right)\right)\\&\quad\leq\frac{f\left(m\right)\left(M1_K-\Phi\left(A\right)\right)+f\left(M\right)\left(\Phi\left(A\right)-m1_K\right)}{M-m}-f\left(\Phi\left(A\right)\right).\end{aligned} \tag{9}$$

Since $m1_K\leq\Phi\left(A\right)\leq M1_K$, then by using (4) for $a=m$, $b=M$ and the continuous functional calculus, we have

$$\begin{aligned}&\frac{f\left(m\right)\left(M1_K-\Phi\left(A\right)\right)+f\left(M\right)\left(\Phi\left(A\right)-m1_K\right)}{M-m}-f\left(\Phi\left(A\right)\right)\\&\leq\frac{f'_-\left(M\right)-f'_+\left(m\right)}{M-m}\left(M1_K-\Phi\left(A\right)\right)\left(\Phi\left(A\right)-m1_K\right)\\&\leq\frac{1}{4}\left(M-m\right)\left[f'_-\left(M\right)-f'_+\left(m\right)\right]1_K.\end{aligned} \tag{10}$$

By making use of (9) and (10) we get the desired result (5). □

Corollary 1. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum* $\operatorname{Sp}(A)\subset[m,M]$.

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$
\begin{aligned}
0 &\le \Phi(f(A)) - f(\Phi(A)) \\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}(M1_K-\Phi(A))(\Phi(A)-m1_K) \\
&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_K.
\end{aligned}
\tag{11}
$$

We also have the following scalar inequality of interest:

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $t\in[0,1]$, then*

$$
\begin{aligned}
&2\min\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \\
&\le (1-t)f(a)+tf(b)-f((1-t)a+tb) \\
&\le 2\max\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right].
\end{aligned}
\tag{12}
$$

The proof follows, for instance, by Corollary 1 from [5] for $n=2$, $p_1=1-t$, $p_2=t$, $t\in[0,1]$ and $x_1=a$, $x_2=b$.

Theorem 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum* $\operatorname{Sp}(A)\subset[m,M]$.

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_K-\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right) \\
&\le \frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)) \\
&\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_K-\Phi\left(\left|A-\frac{1}{2}(m+M)1_K\right|\right)\right) \\
&\le \frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-\Phi(f(A))
\end{aligned}
\tag{14}
$$

$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_K+\Phi\left(\left|A-\frac{1}{2}(m+M)1_H\right|\right)\right).$$

Proof. We have from (12) that

$$\begin{aligned}(15)\qquad & 2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\ & \leq (1-t)f(m)+tf(M)-f((1-t)m+tM) \\ & \leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],\end{aligned}$$

for all $t\in[0,1]$.

Utilizing the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ we get from (15) that

$$\begin{aligned}(16)\qquad & 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \\ & \leq (1-T)f(m)+Tf(M)-f((1-T)m+TM) \\ & \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),\end{aligned}$$

in the operator order.

If we take in (16)

$$0\leq T=\frac{A-m1_H}{M-m}\leq 1_H,$$

then, like in the proof of Theorem 2, we get

$$\begin{aligned}(17)\qquad & 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\ & \quad\times\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right) \\ & \leq \frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A) \\ & \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\ & \quad\times\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right).\end{aligned}$$

Since $m1_K\leq\Phi(A)\leq M1_K$, then by writing the inequality (17) for $\Phi(A)$ instead of A we get (13).

If we take Φ in (17), then we get

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\Phi\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right) \\
&\leq\Phi\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]-\Phi f(A) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\Phi\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right),
\end{aligned}
$$

which is equivalent to (14). □

Corollary 2. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$
\begin{aligned}
(18)\qquad &2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_K-\Phi\left(\left|A-\frac{1}{2}(m+M)1_H\right|\right)\right) \\
&\leq\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-\Phi(f(A)) \\
&\leq\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right).
\end{aligned}
$$

We also have:

Corollary 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$
\begin{aligned}
(19)\qquad \Phi(f(A))-f(\Phi(A)) &\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right) \\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_K.
\end{aligned}
$$

Proof. From (9) we have

$$\Phi(f(A)) - f(\Phi(A)) \le \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M-m} - f(\Phi(A))$$

and from (14) we have

$$\frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M-m} - f(\Phi(A)) \le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \times \left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right),$$

which produce the desired result (19). □

Remark 1. If $f:[m,M]\to\mathbb{R}$ is an operator convex function on $[m,M]$, A a selfadjoint operator with the spectrum $\mathrm{Sp}(A)\subset[m,M]$ and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}0 &\le \Phi(f(A)) - f(\Phi(A)) \\ &\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \times \left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right) \\ &\le 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_K.\end{aligned} \tag{20}$$

We also have [4]:

Lemma 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K-Lipschitzian on $[a,b]$, then*

$$\begin{aligned}&|(1-t)f(a) + tf(b) - f((1-t)a+tb)| \\ &\le \frac{1}{2}K(b-t)(t-a) \le \frac{1}{8}K(b-a)^2\end{aligned} \tag{21}$$

for all $t\in[0,1]$.

The constants $1/2$ and $1/8$ are the best possible in (21).

Remark 2. If $f:[a,b]\to\mathbb{R}$ is twice differentiable and $f''\in L_\infty[a,b]$, then

$$\begin{aligned}&|(1-t)f(a) + tf(b) - f((1-t)a+tb)| \\ &\le \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a) \le \frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2,\end{aligned} \tag{22}$$

where $\|f''\|_{[a,b],\infty} := \mathrm{essup}_{t\in[a,b]}|f''(t)| < \infty$. The constants $1/2$ and $1/8$ are the best possible in (22).

We have:

Theorem 4. *Let* $f:[m,M]\to\mathbb{R}$ *be a twice differentiable convex function on* $[m,M]$ *with* $\|f''\|_{[m,M],\infty}:=\operatorname{essup}_{t\in[m,M]}f''(t)<\infty$ *and* A *a selfadjoint operator with the spectrum* $\operatorname{Sp}(A)\subset[m,M]$*. If* $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$*, then*

$$
\begin{aligned}
&\Phi(f(A))-f(\Phi(A)) \\
&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K) \\
&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.
\end{aligned}
\tag{23}
$$

Proof. From (22) and the continuous functional calculus, we get

$$
\begin{aligned}
0&\le\frac{f(m)(M1_H-B)+f(M)(B-m1_H)}{M-m}-f(B) \\
&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-B)(B-m1_H) \\
&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H
\end{aligned}
\tag{24}
$$

where B is a selfadjoint operator with the spectrum $\operatorname{Sp}(B)\subset[m,M]$.

If we use (24) for $\Phi(A)$ we get

$$
\begin{aligned}
0&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)) \\
&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K) \\
&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.
\end{aligned}
\tag{25}
$$

Since

$$
\begin{aligned}
&\Phi(f(A))-f(\Phi(A)) \\
&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)),
\end{aligned}
$$

hence by (25) we get (23). □

Corollary 4. *Let* $f:[m,M]\to\mathbb{R}$ *be an operator convex function on* $[m,M]$ *and* A *a selfadjoint operator with the spectrum* $\operatorname{Sp}(A)\subset[m,M]$*.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$*, then*

$$
\begin{aligned}
0&\le\Phi(f(A))-f(\Phi(A)) \\
&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K) \\
&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.
\end{aligned}
\tag{26}
$$

3. Some examples

We consider the exponential function $f(x)=\exp(\alpha x)$ with $\alpha\in\mathbb{R}\setminus\{0\}$. This function is convex but not operator convex on $\mathbb{R}$. If A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $m<M$ and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then by (5), (19) and (23) we have

$$\begin{aligned}
&\Phi(\exp(\alpha A))-\exp(\alpha\Phi(A))\\
&\le\alpha\frac{\exp(\alpha M)-\exp(\alpha m)}{M-m}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\
&\le\frac{1}{4}\alpha(M-m)[\exp(\alpha M)-\exp(\alpha m)]1_K,
\end{aligned}\tag{27}$$

$$\begin{aligned}
&\Phi(\exp(\alpha A))-\exp(\alpha\Phi(A))\\
&\le 2\left[\frac{\exp(\alpha m)+f(\alpha M)}{2}-\exp\left(\alpha\frac{m+M}{2}\right)\right]\\
&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\\
&\le 2(M-m)\left[\frac{\exp(\alpha m)+f(\alpha M)}{2}-\exp\left(\alpha\frac{m+M}{2}\right)\right]1_K
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
&\Phi(\exp(\alpha A))-\exp(\alpha\Phi(A))\\
&\le\frac{1}{2}\alpha^2\begin{cases}\exp(\alpha M) & \text{if } \alpha>0\\ \\ \exp(\alpha m) & \text{if } \alpha<0\end{cases}\times(M1_K-\Phi(A))(\Phi(A)-m1_K)\\
&\le\frac{1}{8}\alpha^2(M-m)^2\begin{cases}\exp(\alpha M) & \text{if } \alpha>0\\ \\ \exp(\alpha m) & \text{if } \alpha<0\end{cases}\times 1_K.
\end{aligned}\tag{29}$$

The function $f(x)=-\ln x$, $x>0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $0<m<M$ and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then by (11), (20) and (26) we have

$$\begin{aligned}
&0\le\ln(\Phi(A))-\Phi(\ln(A))\\
&\le\frac{1}{mM}(M1_V-\Phi(A))(\Phi(A)-m1_K)\le\frac{1}{4mM}(M-m)^2 1_K,
\end{aligned}\tag{30}$$

$$\begin{aligned}
&0\le\ln(\Phi(A))-\Phi(\ln(A))\\
&\le 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\\
&\le 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right)1_K
\end{aligned}\tag{31}$$

and

$$0 \le \ln(\Phi(A)) - \Phi(\ln(A)) \le \frac{1}{2m^2}(M1_K - \Phi(A))(\Phi(A) - m1_K) \le \frac{1}{8m^2}(M-m)^2 1_K. \tag{32}$$

We observe that if $M > 2m$ then the bound in (30) is better than the one from (32). If $M < 2m$, then the conclusion is the other way around.

The function $f(x) = x\ln x$, $x > 0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (11), (20) and (26) we have

$$0 \le \Phi(A\ln(A)) - \Phi(A)\ln(\Phi(A)) \le \frac{\ln(M) - \ln(m)}{M-m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \le \frac{1}{4}(M-m)[\ln(M) - \ln(m)]1_K, \tag{33}$$

$$\begin{aligned} &0 \le \Phi(A\ln(A)) - \Phi(A)\ln(\Phi(A)) \\ &\le 2\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\ &\quad \times \left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right) \\ &\le 2(M-m)\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]1_K \end{aligned} \tag{34}$$

and

$$0 \le \Phi(A\ln(A)) - \Phi(A)\ln(\Phi(A)) \le \frac{1}{2m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \le \frac{1}{8m}(M-m)^2 1_K. \tag{35}$$

Consider the power function $f(x) = x^r$, $x \in (0,\infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1,\infty)$, then f is convex and for $r \in [-1, 0] \cup [1, 2]$ is operator convex. If we use the inequalities (5), (19) and (23) we have for $r \in (-\infty, 0] \cup [1,\infty)$ that

$$\Phi(A^r) - (\Phi(A))^r \le r\frac{M^{r-1} - m^{r-1}}{M-m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \le \frac{1}{4}r(M-m)\left(M^{r-1} - m^{r-1}\right)1_K, \tag{36}$$

$$\begin{aligned}(37)\qquad &\Phi\left(A^{r}\right)-(\Phi(A))^{r}\\ &\leq 2\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right)^{r}\right]\\ &\times\left(\frac{1}{2}(M-m)1_{K}+\left|\Phi(A)-\frac{1}{2}(m+M)1_{K}\right|\right)\\ &\leq 2(M-m)\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right)^{r}\right]1_{K}\end{aligned}$$

and

$$\begin{aligned}(38)\qquad &\Phi\left(A^{r}\right)-(\Phi(A))^{r}\\ &\leq\frac{1}{2}r(r-1)\begin{cases}M^{r-2}\text{ for }r\geq 2\\ m^{r-2}\text{ for }r\in(-\infty,0]\cup[1,2)\end{cases}\\ &\times\left(M1_{K}-\Phi(A)\right)\left(\Phi(A)-m1_{K}\right)\\ &\leq\frac{1}{8}r(r-1)(M-m)^{2}\begin{cases}M^{r-2}\text{ for }r\geq 2\\ m^{r-2}\text{ for }r\in(-\infty,0]\cup[1,2)\end{cases}\times 1_{K},\end{aligned}$$

where A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $0<m<M$ and $\Phi\in\mathfrak{P}_{N}[\mathcal{B}(H),\mathcal{B}(K)]$.

If $r\in[-1,0]\cup[1,2]$, then we also have $0\leq\Phi\left(A^{r}\right)-(\Phi(A))^{r}$ in the inequalities (36)-(38).

For $r=-1$ we have the inequalities

$$\begin{aligned}(39)\qquad &0\leq\Phi\left(A^{-1}\right)-(\Phi(A))^{-1}\\ &\leq\frac{M+m}{M^{2}m^{2}}\left(M1_{K}-\Phi(A)\right)\left(\Phi(A)-m1_{K}\right)\\ &\leq\frac{1}{4}(M-m)^{2}\frac{M+m}{M^{2}m^{2}}1_{K},\end{aligned}$$

$$\begin{aligned}(40)\qquad &0\leq\Phi\left(A^{-1}\right)-(\Phi(A))^{-1}\\ &\leq\frac{(M-m)^{2}}{mM(m+M)}\left(\frac{1}{2}(M-m)1_{K}+\left|\Phi(A)-\frac{1}{2}(m+M)1_{K}\right|\right)\\ &\leq\frac{(M-m)^{3}}{mM(m+M)}1_{K}\end{aligned}$$

and

$$\begin{aligned}(41)\qquad &0\leq\Phi\left(A^{-1}\right)-(\Phi(A))^{-1}\\ &\leq\frac{1}{m^{3}}\left(M1_{K}-\Phi(A)\right)\left(\Phi(A)-m1_{K}\right)\leq\frac{1}{4m^{3}}(M-m)^{2}1_{K},\end{aligned}$$

where A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $0<m<M$ and $\Phi\in\mathfrak{P}_{N}[\mathcal{B}(H),\mathcal{B}(K)]$.

4. Conclusion

In this paper we obtained several operator inequalities providing upper bounds for the celebrated Davis-Choi-Jensen's Difference for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\mathrm{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for fundamental convex and operator convex functions of interest, to ilustrate the main results, were also provided.

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